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CONCERNING DU BOIS-REYMOND'S TWO RELATIVE INTEGRABILITY THEOREMS.

By Eliakim Hastings Moore.

1. Introduction. The two theorems of du Bois-Reymond to be considered are suggestively (and incompletely) expressed in the form:

I. *A continuous function of (properly) integrable functions is integrable.*

II. *An integrable function of an integrable function is integrable.*

Du Bois-Reymond announced the first theorem in 1880 (*Math. Annalen*, vol. 16, p. 112), and published a proof two years later (*ibid.*, vol. 20, pp. 122–124), and in the latter connection he announced the second theorem.

I have found no further reference to the second theorem; in §5 I shall show by a simple example that it is not true.

When one thinks, in the first theorem, of the integral of the compound function as the limit of a sum, I shall show in §§2, 3, 4, that the range of values of the terms of the sum may be extended without alteration of the integral-limit.

In the case of the product of two integrable functions (for which case the original theorem was proved by du Bois-Reymond in 1875, in vol. 75 of the *Journal für Math.*, p. 24), this extension was made by Dini (1878: Dini-Lüroth, *Functionen einer reellen Veränderlichen*, §190, p. 347); his process of proof is not applicable to the general case. Du Bois-Reymond's general proof (1882) is however capable of immediate extension. I give the proof of the theorem of wider integrability and of the uniformity of this integrability for the set of all subintervals of the interval of integration by a process somewhat different from du Bois-Reymond's process and in a desirably explicit form.

This notion of wider integrability I have found of use in effecting a certain extension of the second mean value theorem of the integral calculus.

2. Du Bois-Reymond's First Integrability Theorem. *If the function*

$$F(y_1, \ldots, y_m)$$

is a continuous function of its m arguments $y_1, \ldots, y_m$ at all points $Y = (y_1, \ldots, y_m)$ of the region R:

(1) $\alpha_1 \leqq y_1 \leqq \beta_1, \ldots, \alpha_g \leqq y_g \leqq \beta_g, \ldots, \alpha_m \leqq y_m \leqq \beta_m$:

if, furthermore, the functions $\phi_1(x), \ldots, \phi_m(x)$ *are properly integrable on the x-interval* $a \ldots b$ *and take on (only) values which satisfy the conditions:*

$$(2) \qquad a_1 \leqq \phi_1(x) \leqq \beta_1, \ldots, a_m \leqq \phi_m(x) \leqq \beta_m \qquad (a \leqq x \leqq b);$$

then, the function

$$f(x) = F(\phi_1(x), \ldots, \phi_m(x))$$

is properly integrable on the x-interval $a \ldots b$.

3. Preliminaries of the Extension. The function $F(y_1, \ldots, y_m)$ is continuous on the closed region R, and hence on that region it has definite upper and lower limits (or bounds):

$$\overline{F}_R = \overline{\mathrm{L}}_{Y \mid R} F(y_1, \ldots, y_m), \qquad \underline{F}_R = \underline{\mathrm{L}}_{Y \mid R} F(y_1, \ldots, y_m),$$

and it is uniformly continuous; that is, for every ϵ a δ_ϵ exists such that

$$\left| F(y_1', \ldots, y_m') - F(y_1'', \ldots, y_m'') \right| < \epsilon$$

for every two points Y', Y'' of the region R for which

$$\left| y_1' - y_1'' \right| < \delta_\epsilon, \ldots, \left| y_m' - y_m'' \right| < \delta_\epsilon.$$

The function $f(x)$ is, on the interval $a \ldots b$, a single valued function with definite upper and lower bounds $\overline{f}_{ab}$, $\underline{f}_{ab}$ which lie on the interval $\underline{F}_R \ldots \overline{F}_R$.

We consider a partition π of the interval $a \ldots b$ into a finite number (n) of intervals* $x_0 x_1, \ldots, x_{n-1} x_n (x_0 = a, x_n = b)$ and the corresponding general sum

$$f_\pi = f_1 \delta_1 + \ldots + f_n \delta_n,$$

where $\delta_k = x_k - x_{k-1}$ and f_k is any value lying on the interval $\underline{f}_k \ldots \overline{f}_k$ bounded by the upper and lower limits $\overline{f}_k$, $\underline{f}_k$ of $f(x)$ on the x-interval $x_{k-1} \ldots x_k$.

Du Bois-Reymond's theorem affirms the existence of the limit:

$$\mathrm{L}_{\delta_k = 0} f_\pi,$$

in the usual sense of the theory of proper definite integrals.

Now we have as definition

$$f(x) = F(\phi_1(x), \ldots, \phi_m(x)),$$

* For brevity, here and below in several limit formulas an interval $a' \ldots b'$ is denoted by $a'b'$.

and so

$$\bar{f}_k = \bar{\mathbf{L}}_{x \mid x_{k-1}x_k} F(\phi_1(x), \ldots, \phi_m x)), \qquad \underline{f}_k = \underline{\mathbf{L}}_{x \mid x_{k-1}x_k} F(\phi_1(x), \ldots, \phi_m(x)).$$

We consider the more inclusive or wider limits

$$\bar{f}_k^* = \bar{\mathbf{L}}\, F(\phi_{1k}, \ldots, \phi_{mk}), \qquad \underline{f}_k^* = \underline{\mathbf{L}}\, F(\phi_{1k}, \ldots, \phi_{mk}),$$

that is, the upper and lower limits of the set of all values of $F(\phi_{1k}, \ldots, \phi_{mk})$ obtained by allowing each argument ϕ_{gk} $(g = 1, \ldots, m)$ independently to run through the interval $\underline{\phi}_{gk} \ldots \bar{\phi}_{gk}$ bounded by the upper and lower limits of $\phi_g(x)$ on the x-interval $x_{k-1} \ldots x_k$; and we denote by f_k^* any number of the interval $\underline{f}_k^* \ldots \bar{f}_k^*$ and consider the wider general sum:

$$f_\pi^* = f_1^* \delta_1 + \ldots + f_n^* \delta_n.$$

Thus every partition π of $a \ldots b$ determines such sums f_π^* connected with $a \ldots b$, and the theorem of the wider integrability affirms the existence of the limit in like sense of these wider general sums and the relation

$$\mathbf{L}_{\delta_k = 0} f_\pi^* = \mathbf{L}_{\delta_k = 0} f_\pi.$$

The partition π of $a \ldots b$ effects a partition π' of a subinterval $a' \ldots b'$ of $a \ldots b$ into (at most) n partial intervals of lengths $\delta_1', \ldots, \delta_n'$ where then $0 \leqq \delta_k' \leqq \delta_k$, $\delta_1' + \ldots + \delta_n' = b' - a'$. We consider the general sum

$$f_{a'\pi}^{b'} = f_1 \delta_1' + \cdots + f_n \delta_n'$$

and the wider general sum

$$f_{a'\pi}^{b'*} = f_1^* \delta_1' + \cdots + f_n^* \delta_n'.$$

From these general sums we obtain the corresponding upper sums by replacing throughout f_k, f_k^* by $\bar{f}_k, \bar{f}_k^*$; and similarly we obtain the corresponding lower sums.

In these notations it is evident that

$$f_\pi = f_{a\pi}^b, \quad f_\pi^* = f_{a\pi}^{b*},$$

and that, for a given partition π and interval $a' \ldots b'$ and corresponding partition π' of $a' \ldots b'$,

$$\underline{f}_{a'\pi}^{b'} \leqq \underline{f}_{a'\pi}^{b'*} \leqq \bar{f}_{a'\pi}^{b'} \leqq f^{b'} .$$

and so that every sum $f^{b'}_{a'\pi'}$, is a sum $f^{b'}_{a'\pi}$, while the converse is in general not true.

We have always the relations:

$$0 \leqq \bar{f}^{b'}_{a'\pi'} - \underline{f}^{b'}_{a'\pi'} \leqq \bar{f}^{b'}_{a'\pi} - \underline{f}^{b'}_{a'\pi} \leqq \bar{f}^{b}_{a\pi} - \underline{f}^{b}_{a\pi},$$

with similar relations for the wider sums.

4. The more intense First Integrability Theorem. *The limits*

$$\mathop{\mathbf{L}}_{\delta_k = 0} f^{b'*}_{a'\pi}$$

of the wider sums $f^{b'*}_{a'\pi}$ with respect to the various intervals $a' \,..\, b'$ of $a \,...\, b$ exist, and the convergence is uniform, for the set of all such intervals $a' \,...\, b'$. The limiting values are of course

$$\mathop{\mathbf{L}}_{\delta_k = 0} f^{b'}_{a'\pi'} = \int_{a'}^{b'} f(x)\,dx \qquad (a \leqq a' < b' \leqq b).$$

Remark. The theorem, in so far as it concerns the sums $f^{b'*}_{a'\pi}$ instead of the sums $f^{b'*}_{a'\pi}$, affirms somewhat more than the uniformity of the wider integrability.

Proof. We shall exhibit for a given ϵ a partition π_ϵ of $a \,...\, b$ such that

$$\bar{f}^{b*}_{a\pi_\epsilon} - \underline{f}^{b*}_{a\pi_\epsilon} < \epsilon.$$

From this inequality, in view of the final remark of §3, the theorem follows by considerations similar to certain considerations* of the usual theory of definite integrals.

The m functions $\phi_g(x)$ $(g = 1, \ldots, m)$ are integrable from a to b. Hence, by Riemann's necessary condition, there exists for each function $\phi_g(x)$ with respect to any two given positive numbers σ, λ a partition $\pi_{g\sigma\lambda}$ of $a \,...\, b$ such that the sum of those intervals of $\pi_{g\sigma\lambda}$ on which the oscillation of $\phi_g(x)$ is greater than σ is itself less than λ. By superposition of any m such partitions $\phi_{g\sigma\lambda}$ $(g = 1, \ldots, m)$ we have a partition $\pi_{\sigma\lambda}$ such that on every interval, apart from intervals of total length less than $m\lambda$, the oscillation of each function $\phi_g(x)$ $(g = 1, \ldots, m)$ is at most σ. Taking for σ the $\delta_{\epsilon'}$ connected with the uniform continuity of $F(y_1, \ldots, y_m)$, ϵ' being a positive number subject to

* *Cf.* the theorem of Dini, *l. c.*, §185, p. 330.

later determination, we see that on every interval of $\pi_{\sigma\lambda}$, apart from the intervals specified, the corresponding oscillation $\bar{f}_k - \underline{f}_k$ is less than ϵ' while on every excepted interval the oscillation $\bar{f}_k - \underline{f}_k$ is at most $\bar{F}_R - \underline{F}_R$. Hence for this partition $\pi_{\sigma\lambda}$ $(\sigma = \delta_{\epsilon'})$ we have

$$\bar{f}^{b*}_{a\pi_{\sigma\lambda}} - \underline{f}^{b*}_{a\pi_{\sigma\lambda}} < \epsilon'(b-a) + m\lambda(\bar{F}_R - \underline{F}_R).$$

Thus, if we set, with respect to the given ϵ,

$$\epsilon' = \frac{1}{2}\frac{\epsilon}{b-a}, \qquad \sigma = \delta_{\epsilon'}, \qquad \lambda = \frac{1}{2m}\frac{\epsilon}{\bar{F}_R - \underline{F}_R + 1},$$

we have in a partition $\pi_{\sigma\lambda}$ the desired partition π_ϵ.

5. An Example in Contravention of du Bois-Reymond's Second Integrability Theorem.

The Theorem. It the function $f(x)$ is properly integrable from a to b and has the limits $\bar{f}_{ab}$, $\underline{f}_{ab}$, and if the function $g(y)$ is properly integrable from $\underline{f}_{ab}$ to $\bar{f}_{ab}$, then the compound function $g(f(x))$ is properly integrable from a to b.

The Example. We take the x-interval $0 \ldots 1$ and consider the exhibition of a number x to the base 3:

$$x = \sum_{r=0}^{\infty} i_{r,x} 3^{-r}$$

where each digit $i_{r,x}$ is 0, 1, or 2; we exclude the repetend 2, and thus have a unique exhibition of every number x $(0 \leqq x \leqq 1)$.

The function $f(x)$ for x with a terminating exhibition:

$$x = \sum_{r=0}^{r_0} i_{r,x} 3^{-r}, \qquad (i_{r_0,x} \neq 0)$$

shall have the value

$$f(x) = (-1)^{i_{r_0,x}} 3^{-r_0},$$

while it shall be 0 for all other values of x. This function is properly integrable from 0 to 1, and it has everywhere densely values both positive and negative.

The function $g(y)$:

$$g(y) = \begin{Bmatrix} +1 & y > 0 \\ 0 & y = 0 \\ -1 & y < 0 \end{Bmatrix},$$

is properly integrable on every finite interval.

The function $h(x) = g(f(x))$ for $0 \leqq x \leqq 1$ has the values:

$$h(x) = (-1)^{i_{r_0, x}}, \quad \text{or} \quad 0,$$

according as x has or has not a terminating exhibition. Thus it has values $0, +1, -1$ each everywhere densely. It is accordingly not properly integrable.

THE UNIVERSITY OF CHICAGO, MAY 29, 1901.

ON A THEOREM OF KINEMATICS.

By Paul Saurel.

It is a well known theorem of kinematics that every displacement of a rigid body is equivalent to a rotation followed by a translation parallel to the axis of rotation. The following elementary demonstration of this theorem may be of interest.

We shall assume the following theorems:

I. Every displacement of a plane figure in its plane is equivalent to a translation, or to a rotation about some point of the plane.

II. Every displacement of a rigid body about a fixed point is equivalent to a rotation about an axis passing through that point.

As a corollary to the first of these theorems it follows that every displacement of a rigid body which transforms a system of parallel lines into itself, is equivalent to a rotation about one of the parallel lines followed by a translation in the direction of the lines, or to two translations, one perpendicular, the other parallel to the lines in question.

From the second of the above theorems it may be shown in the usual way that every displacement of a rigid body is equivalent to a translation followed by a rotation. Consider three points of the body, not lying in a right line, which are initially in the positions A_1, B_1, C_1 and finally in the positions A_2, B_2, C_2. A translation of the body will move the first point from the position A_1 to the position A_2, while the second and third points will take the positions B_1', C_1'. From theorem II it follows that these points may then be brought from the positions B_1', C_1' to the positions B_2, C_2, by a rotation about an axis through A_2.

After the translation every line of the body occupies a position parallel to its original position; after the rotation only those lines of the body which were parallel to the axis of rotation occupy positions parallel to their original positions. Thus, if the displacement considered is not equivalent to a translation, there is one and only one system of parallel lines which is transformed into itself, namely, the lines parallel to the axis of rotation.

From the corollary to the first theorem the desired theorem now follows at once: *Every displacement that is not equivalent to a translation is equivalent to a rotation followed by a translation parallel to the axis of rotation.*

Incidentally it may be noticed that, however we decompose the displacement into a translation and a rotation, the axis of rotation is always one of the system of parallel lines which the displacement transforms into itself.

NEW YORK, JANUARY 19, 1901.

THE COLLINEATIONS OF SPACE WHICH TRANSFORM A NON-DEGENERATE QUADRIC SURFACE INTO ITSELF.*

By Ruth G. Wood.

The object of this paper is to discuss the ∞^6 collineations of space which transform a non-degenerate quadric surface into itself. These fall into two classes. The collineations of the *first* kind leave the two systems of generators of the quadric invariant, while the collineations of the *second* kind interchange the two systems. When the quadric surface degenerates into the imaginary circle at infinity, these collineations of the first and second kinds become respectively the displacements and the symmetry transformations of euclidean geometry. We shall then call a collineation of the first kind a non-euclidean displacement, and a collineation of the second kind a non-euclidean symmetry transformation. These transformations for euclidean geometry have been discussed by Mr. Gale in a paper entitled† *Wiener's Theory of Displacements, with Application to the Proof of Four Theorems of Chasles.* The methods here employed are similar to those used by Mr. Gale, while the theorems obtained correspond in non-euclidean space to the four theorems of Chasles proved in his paper.

This correspondence is readily seen if we choose the quadric surface for the absolute. Two lines are then said to be conjugate polars with respect to the absolute when the polar planes of all the points of one line pass through the other. Two planes are said to be conjugate or perpendicular when the pole of one lies in the other. Two intersecting lines are said to be perpendicular when each intersects the conjugate of the other, and a line is said to be perpendicular to a plane when it passes through the pole of the plane.

The fundamental operation in this discussion is the involutory transformation known as a skew reflection on conjugate polars of the quadric. By this operation a point P is transformed into a point P' such that the line PP' is divided harmonically by the two polars, which are called the *directrices* of the skew reflection. When the quadric degenerates into the imaginary circle at infinity the skew reflection becomes reflection in a line, the transformation

* This paper was read before the American Mathematical Society, December 28, 1900.

† Annals of Mathematics, ser. 2, vol. 2 (1900), p. 1.

used by Mr. Gale in his discussion. We shall then call a skew reflection on conjugate polars of the quadric a *non-euclidean line reflection*.

We shall first show that any non-euclidean displacement may be resolved into the product of two non-euclidean line reflections. This affords a simple method of resolving non-euclidean displacements and symmetry transformations so that some of their fundamental properties are at once evident. From these resolutions theorems concerning the displacements of straight lines and plane figures in non-euclidean space are readily deduced, which, as before mentioned, become well known theorems of Chasles when the quadric degenerates into the imaginary circle at infinity.

Non-Euclidean Displacements.

Non-euclidean displacements leave the two systems of generators of a quadric surface invariant and may be classified according to the invariant figure upon the invariant quadric. The resolution into skew reflections depends essentially upon a theorem of Wiener concerning projectivities.

A projectivity upon a line is the projective relation between the points $A, B, C \ldots$ and the points $A', B', C' \ldots$ of that line, and may be denoted by the symbol $ABC \pi A'B'C'$. The two points on the line which correspond to themselves are called double points. If P, Q are corresponding points, and D, D' the double points of a projectivity, the cross ratio (DD', PQ) is constant. When this cross ratio is harmonic the projectivity is said to be an involution. Wiener's theorem above referred to is:

*Any projectivity $DD'P \pi DD'Q$ may be resolved in ∞^1 ways into the product of two involutions.**

Proof. Any point upon PQ may be chosen as a double point D_1 of the first involution. The second double point D_1' is then determined as the fourth harmonic of D, D' and D_1. In this involution the point P will correspond to a point P'. The double points D_2, D_2' of the second involution are now chosen to divide harmonically D, D' and P', Q.

If D and D' fall together in D, so that $D^2P \pi D^2Q$, then D and any point D_1 upon PQ are chosen as the double points of the first involution. In this involution the point P will correspond to a point P'. The double points of the second involution are D_2 and D, D_2 being chosen as the fourth harmonic of $D\, P'$ and Q.

* *Leipziger Berichte*, vol. 43 (1891), p. 648.

This being established we may now state our

Fundamental Theorem: *Any non-euclidean displacement may be resolved into the product of two non-euclidean line reflections.*

We will give the proof for the following five types of displacements, which are evidently all that can occur.

(a) *Two generators of each system remain fixed.** The conjugate polars which are the opposite edges of the tangential tetrahedron formed by the four fixed generators we shall call the axes of this displacement. The displacement depends upon fourteen parameters and is completely determined by the invariant quadric S, the vertices A, B, C, D of the tangential tetrahedron and a pair of corresponding general points $P\ P'$ which lie upon S. Through P and P' draw the lines intersecting the axes AD and BC in L, L' and M, M' respectively. Then upon AD and BC are established projectivities such that $ADL \pi ADL'$ and $BCM \pi BCM'$. Each of these projectivities may be resolved in ∞^1 ways into the product of two involutions. The lines joining in pairs the double points of these involutions are conjugate polars with respect to the quadric and may be taken as the directrices of two non-euclidean line reflections, the product of which is a displacement which leaves the quadric S invariant. Since the double points may be joined in four ways, four different displacements are thus obtained each of which leaves the tetrahedron $ABCD$ invariant and transforms one intersection of the line LM with the quadric into an intersection of the line $L'M'$ with the quadric. Hence one of these displacements transforms the point P into the point P' and is consequently the given displacement. It may be found by at most four trials. The resolution of this displacement into the product of two line reflections may thus be effected in ∞^2 ways.

(b) *Two generators of one system and one of the other remain fixed.*

This displacement has two fixed points, the intersections of the fixed generators. It depends upon thirteen parameters and is completely defined by the quadric S, the two fixed points A, B and a pair of corresponding points P, P' on S. Through P and P' pass the generators which intersect the fixed generator AB in the points L and L'. Then upon AB is established a projectivity $ABL \pi ABL'$ which may be resolved in ∞^1 ways into the product of two involutions with double points D_1, D_1' and D_2, D_2'. Through D_1 and

* The fundamental theorem for case (a) has been stated by Mr. Wilson, *Trans. Amer. Math. Soc.*, vol. 1 (1900), p. 196

D_1' draw any pair of conjugate polars, which may be done in ∞^1 ways. By a reflection on these lines as directrices, the point P is transformed into a point P'' on the quadric. The directrices of the second line reflection are conjugate polars through D_2 and D_2' and must be chosen so as to transform P'' into P'.

This may be done as follows: Through D_2 draw the generator of the first system and note the points M and M' in which generators through P'' and P' intersect this generator. The first directrix of the line reflection is the line joining the fourth harmonic of M, M' and D_2 with D_2'. The second directrix is its conjugate polar and passes through D_2. The resolution of this displacement into the product of two line reflections is thus effected in ∞^2 ways.

(*c*) *One generator of each system remains fixed.*

This displacement has one fixed point A, the intersection of the fixed generators. It depends upon twelve parameters and is completely determined by the quadric S, the fixed point A and a pair of corresponding points P, P' on S. Through P and P' draw generators which will intersect one of the fixed generators in L and L'. Then upon this generator is established a projectivity such that $A^2L \,\pi\, A^2L'$. This projectivity may be resolved in ∞^1 ways into the product of two involutions with double points A, D_1 and A, D_2. The directrices of the line reflections are now determined as in case (*b*).

The projectivity may also be established upon the other fixed generator. The resolution of this displacement into the product of two line reflections may thus be effected in ∞^2 ways.

(*d*) *Two generators of one system and all of the second system remain fixed.*

All the points of the two fixed generators of the first system are invariant under this displacement which depends upon nine parameters. The displacement is completely determined by the quadric S, the fixed generators, and a pair of corresponding points which will lie upon a generator of S belonging to the second system. Then upon this generator is established a projectivity in which the intersections of the two fixed generators of the first system are double points. This projectivity may be resolved in ∞^1 ways into the product of two involutions with double points D_1, D_1' and D_2, D_2'. The directrices of the first line reflection are any pair of conjugate polars through D_1 and D_1'. They will be the opposite edges of a tangential tetrahedron formed by the generators through D_1 and D_1' and another generator of the second system. The directrices of the second line reflection are then uniquely determined. They are the conjugate polars which are opposite edges of a tangential

tetrahedron formed by the generators through D_2 and D_2' and the same generator of the second system. The resolution of this displacement into the product of two line reflections is thus effected in ∞^2 ways.

(*e*) *One generator of one system and all of the second system remain fixed.*

The points of the fixed generator of the first system remain invariant under this displacement which depends upon eight parameters. The displacement is completely determined by the quadric S, the fixed generator, and a pair of corresponding points which will lie upon a generator belonging to the second system. Then upon this generator is established a projectivity whose double points fall together in the intersection A with the fixed generator of the first system. This projectivity may be resolved in ∞^1 ways into the product of two involutions with double points D_1, A and D_2, A. The directrices of the line reflections are now determined as in case (*d*). The resolution of this displacement into the product of two line reflections may thus be effected in ∞^2 ways.

Composition of Two Non-Euclidean Displacements.

The resolution of non-euclidean displacements into the product of two non-euclidean line reflections affords a simple method of compounding two such displacements C_1 and C_2. For this it is necessary and sufficient that C_1 and C_2 be resolved so that a line reflection (nn') shall be common to both displacements. We may resolve two displacements C_1 and C_2 of type (*a*) into the product of two line reflections such that the unique transversals nn' of the axes of C_1 and C_2 are the directrices of a line reflection common to both displacements.* Then if $C_1 = (ll')(nn')$ and $C_2 = (nn')(mm')$, $C_1C_2 = (ll')(nn')(nn')(mm') = (ll')(mm')$. When the transversals n and n' fall together the composition is evidently impossible. If C_1 belongs to type (*a*) and C_2 to type (*b*), the directrices (nn') of the line reflection common to both displacements are determined as follows: Through the fixed points of C_1 draw the generators which intersect the single fixed generator AB in L and L'. Find the points on this generator which divide harmonically L, L' and A, B. n and n' are the lines passing through these points and intersecting the axes of C_1. They are easily seen to be conjugate polars and are uniquely determined.

* The unique transversals of two pairs of conjugate polars are conjugate polars, for if a line cut two pairs of conjugate polars the same is true of its conjugate polar.

The determination of the common line reflection for all the different cases presents no difficulties.

If in a non-euclidean displacement $C_1 = (ll')(mm')$, l intersects m, l' will intersect m'. C_1 has then an axis of fixed points which is the line joining the intersections of l with m and l' with m'. Conversely, every displacement having a fixed axis R may be resolved in ∞^2 ways into the product of two line reflections whose directrices intersect in pairs on the fixed axis. This follows at once from the fundamental theorem, if one notices that the corresponding points in the projectivity established upon R fall together as do the two sets of double points of the two involutions of which this projectivity is the product. Hence the directrices of the two line reflections intersect in pairs on R. Such a displacement will be called a *non-euclidean rotation* about the fixed axis. One sees readily that corresponding points lie in a plane with R', the conjugate polar of R.

Resolutions of Non-Euclidean Displacements.

By means of a non-euclidean rotation and our fundamental theorem we readily obtain the following resolutions of non-euclidean displacements.

A straight line L being given every non-euclidean displacement may be resolved into the product of a non-euclidean rotation about L followed by a second non-euclidean rotation.

To prove this, resolve the displacement into the product of two line reflections $(ll')(mm')$ such that l shall be perpendicular to L. Let (nn') be a line reflection such that n is perpendicular to L at its point of intersection with l and let n also intersect m. Then $(ll')(nn')(nn')(mm') = (ll')(mm')$, and $(ll')(nn')$ is a rotation about L while $(nn')(mm')$ is a rotation about an axis R_1 which joins the intersections of n with m and n' with m'.

Let now (nn') be a line reflection such that n is perpendicular to L at its point of intersection with l and let n also intersect m'. Then $(ll')(nn')(nn')(mm') = (ll')(mm')$, and $(ll')(nn')$ is a rotation about L while $(nn')(mm')$ is a rotation about an axis R_2 which joins the intersections of n with m' and n' with m.

In these two resolutions the axis L is perfectly arbitrary and as far as the points in L are concerned the transformation is that of a non-euclidean rotation about R_1 or R_2. Hence

Every displacement of a straight line in non-euclidean space may be effected by a non-euclidean rotation about R_1 or about R_2.

Non-Euclidean Symmetry Transformations.

To obtain corresponding theorems concerning the displacement of plane figures in non-euclidean space it is necessary to study non-euclidean symmetry transformations and in particular that known as a point plane reflection or a non-euclidean plane reflection. By this transformation a point P is transformed into a point P' such that the line PP' is divided harmonically by a point and its polar plane.*

The product of two non-euclidean plane reflections is a non-euclidean displacement having a fixed axis, the intersection of the two planes, and is therefore a non-euclidean rotation. The product of two plane reflections on perpendicular planes S_1 and S_2 passing through a line l is a non-euclidean line reflection having for directrices l and its conjugate l'. For if P is any point in space and P' the point obtained by reflection on S_1 and S_2, one may readily show that PP' is divided harmonically by l and l'. Conversely, every line reflection may be resolved into the product of two plane reflections on perpendicular planes S_1 and S_2. For if P and P' are corresponding points of the line reflection (ll'), pass a plane S_1 through l and P which intersects l' in M. S_2 is now chosen as the polar plane of M and P is transformed into P' by $(S_1)(S_2)$. Two plane reflections on perpendicular planes are commutative, for $(S_1)(S_2)(S_1)(S_2) = (ll')(ll') = 1$ and hence $(S_1)(S_2) = (S_2)(S_1)$.

Since a non-euclidean symmetry transformation interchanges the two systems of generators of the quadric, it is evident that *every non-euclidean displacement may be compounded of a non-euclidean plane reflection on an arbitrary plane followed by a non-euclidean symmetry transformation.*

Our fundamental theorem now enables us to obtain a form for symmetry transformations such that some of their important properties are at once evident.

Consider a plane reflection (S) followed by any non-euclidean symmetry transformation (Σ). The result is a displacement which may be compounded of two line reflections $(ll')(mm')$ such that l shall lie in the plane S. Replace (ll') by a plane reflection in S followed by a plane reflection in the perpendicular plane S_1 through l. Then $(S)(\Sigma) = (ll')(mm') = (S)(S_1)(mm')$.

* It is readily seen that this transformation interchanges the two systems of generators of the quadric surface, for the tangent planes through the pole contain a generator of each system which are interchanged by the transformation.

Whence $(\Sigma) = (S_1)(mm')$ and $S_1(mm')$ is any symmetry transformation. Hence

Every non-euclidean symmetry transformation may be effected by a non-euclidean plane reflection followed by a non-euclidean line reflection.

Replace now (mm') by two plane reflections on perpendicular planes S_2 and S_3 through m such that S_1 and S_3 are perpendicular. Then $S_1(mm') = (S_1)(S_2)(S_3) = (S_3)(S_1)(S_2) = (R)(S_3) = (S_3)(R)$ where $(S_1)(S_2)$ is a rotation about an axis R perpendicular to S_3. Since (R) and (S_3) are commutative it follows that

Every non-euclidean symmetry transformation may be effected by combining in either order a non-euclidean plane reflection followed by a non-euclidean rotation about an axis perpendicular to the plane of the reflection.

The line reflection (mm') may also be replaced by two plane reflections on perpendicular planes S_4 and S_5 through m' such that S_1 and S_5 are perpendicular. Then, since planes through conjugate polars are perpendicular, the planes S_2, S_3, S_4, S_5 are mutually perpendicular and are the four planes of a polar tetrahedron. Since the plane S_1 passes through the pole of S_3 and S_5 the intersection of S_1 and S_2 is the same as that of S_1 and S_4. Then $S_1(mm') = (S_1)(S_4)(S_5) = (R)(S_5)$. Hence

Every non-euclidean symmetry transformation may be resolved into the product of a non-euclidean plane reflection on one of two perpendicular planes followed by a non-euclidean rotation about an axis perpendicular to these planes.

Every symmetry transformation $(R)(S_3) = (R)(S_5)$ has therefore two invariant perpendicular planes S_3 and S_5 and two invariant points on R, namely the poles of S_3 and S_5.

In the resolution of a non-euclidean displacement into the product of a plane reflection followed by a symmetry transformation, the plane of reflection is perfectly arbitrary, and as far as the points in this plane are concerned the transformation is that of a symmetry transformation. Hence

Every displacement of a plane figure in non-euclidean space may be effected by a non-euclidean plane reflection followed by a non-euclidean rotation about an axis perpendicular to the plane of the reflection.

Metric Properties.

Let us now introduce non-euclidean distance and angle. We shall thus see that the correspondence between the transformations discussed here and

those of euclidean space is complete. When the quadric is chosen for the absolute, the angle between two intersecting lines P and Q is given by

$$C \log (PQ, T_1 T_2)$$

where C is an arbitrary constant and T_1 and T_2 are the tangents drawn from the point of intersection to the conic which is cut from the quadric by the plane of P and Q. The distance between two points P and P' is given by

$$C' \log (PP', I_1 I_2)$$

where I_1 and I_2 are the points of intersection of PP' with the quadric.

If H_1 and H_2 are the foci of the involution defined by two pairs of corresponding points PP' and $I_1 I_2$ of a non-euclidean transformation then $PI_1 H_1 H_2 \,\pi\, P'I_2 H_1 H_2$ and hence

$$(PH_1, I_1 I_2) = (P'H_1\, I_2 I_1) = (H_1 P', I_1 I_2)$$

and

$$(PH_2, I_1\, I_2) = (P'H_2, I_2 I_1) = (H_2 P', I_1 I_2)$$

That is, the foci H_1 and H_2 are the middle points of the line PP'. Thus by a non-euclidean line reflection or plane reflection a point P is transformed into a point P' such that P and P' are equidistant from two fixed points. The fixed points in the former case are the intersections of PP' with the directrices of the line reflection, in the latter case the pole and the intersection of PP' with the plane of the reflection.

It is readily seen that any line intersecting the axis at right angles is carried by a given non-euclidean rotation through a constant angle. For the line l is transformed by the rotation $R = (ll')(mm')$ into a line n also perpendicular to the axis R. Then if T_1 and T_2 are the tangents drawn from the intersection of l and m to the conic which is cut from the quadric by the plane of l and m, the cross ratio $(ln, T_1\, T_2)$ remains constant. For the lines T_1 and T_2 will touch the quadric at the points of intersection of R', the conjugate of R, with the quadric. Then upon R' is established a projectivity in which the intersection of R' with l and n are corresponding points and its intersection with T_1 and T_2 double points. Hence the cross ratio $(ln, T_1\, T_2)$ is constant and therefore the angle between l and n is constant. This constant angle is equal to twice the angle between l and m for by (mm') l is carried into a line n and m remains fixed. The cross ratio $(lm, T_1\, T_2)$ becomes $(nm, T_2\, T_1)$: that is, the angle between m and n equals the angle between l and m. Hence the

path curves of a rotation are circles in the non-euclidean sense in planes perpendicular to the axis of rotation.

Resulting Theorems.

To obtain theorems concerning the displacement of a straight line in non-euclidean space, consider the rotations $R_1 = (ll')(mm')$ and $R_2 = (nn')(ss')$ by means of which this displacement may be effected. If P and P' are corresponding points of these rotations then the middle points of PP' will lie upon R'_1 and R'_2 the conjugate polars of R_1 and R_2. For draw l and n through P. Then P remains fixed by (ll') or (nn') and is transformed into P' by (mm') or (ss'). The middle points of P,P' are the intersections of m and m' (or s and s') with PP'. These two middle points coincide with the intersections of PP' with R'_1 and R'_2, since PP', R'_1 and R'_2 are in the same plane and m' (or s') intersects both PP' and R'_1 (or PP' and R'_2). Hence

Theorem I. *If a straight line be displaced in non-euclidean space the loci of the two middle points of chords joining congruent points are straight lines.*

Also since the polar planes of the points on a line pass through its conjugate polar we have

Theorem II. *Planes perpendicular to these chords at their middle points intersect, each set in a straight line.*

The middle points will lie on the conjugate lines of the axes of the two rotations by which the displacement may be effected and the perpendicular planes will pass each set through those axes.

Similarly we obtain theorems concerning the displacement of a plane figure in non-euclidean space by considering the symmetry transformation $(S_1)(S_2)(S_3) = (S_1)(S_4)(S_5)$ by means of which the displacement may be effected. If P and P' are corresponding points in $(S_1)(S_2)(S_3)$ then one middle point of PP' will lie in S_3. For pass S_1 through P, then P remains fixed by (S_1) and goes into P' by $(S_2)(S_3)$. But $(S_2)(S_3)$ is a line reflection and hence one of the middle points lies in S_3. In the same way we see that the other middle point lies in S_5. Hence the theorems.

Theorem III. *If a plane figure be displaced in non-euclidean space the loci of the middle points of chords joining congruent points are perpendicular planes.*

Theorem IV. *Planes perpendicular to three chords at their middle points pass each set through a point.*

The middle points will lie in the invariant planes of the symmetry transformation by means of which the displacement may be effected and the perpendicular planes at these points will pass through the invariant points of the transformation.

For elliptic space, in which the imaginary surface defined by the real equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ is taken for the absolute, the above four theorems hold without modification.

For hyperbolic space, in which the real surface with imaginary generators $x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0$ is taken for the absolute, only one set of middle points lies inside the absolute. These points are all real since the fundamental involutions are hyperbolic. The four theorems are so far modified for real figures that only one middle point and one locus appear in each theorem.

If the absolute degenerates into the imaginary circle at infinity one set of middle points recedes to infinity and the theorems become the four well known theorems of Chasles for euclidean space which are the same as those for hyperbolic space.

YALE UNIVERSITY, FEBRUARY, 1901.

NOTE ON MULTIPLY PERFECT NUMBERS.*

By Jacob Westlund.

In the Annals of Mathematics, ser. 2, vol. 2 (Jan. 1901), p. 103, Dr. D. N. Lehmer proves that no multiply perfect numbers of multiplicity 3, containing less than three distinct primes, exist. The object of the present note is to determine all numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2^{a_2} p_3$ where p_1, p_2, p_3 are three distinct primes and $p_1 < p_2 < p_3$.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity, we have†

$$(1) \qquad 3 = \frac{p_1^{a_1+1} - 1}{p_1^{a_1}(p_1 - 1)} \cdot \frac{p_2^{a_2+1} - 1}{p_2^{a_2}(p_2 - 1)} \cdot \frac{p_3^2 - 1}{p_3(p_3 - 1)}.$$

In order to obtain necessary conditions which p_1 and p_2 must satisfy, we use the inequality†

$$(2) \qquad 3 < \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdot \frac{p_3}{p_3 - 1}.$$

From this we infer that p_1 and p_2 must be 2 and 3 respectively, for the maximum value of $\prod_{n=1}^{3} \frac{p_n}{p_n - 1}$ will exceed 3 only for $p_1 = 2$ and $p_2 = 3$. Hence we have

$$(3) \qquad 3 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{3^{a_2+1} - 1}{3^{a_2} \cdot 2} \cdot \frac{p_3 + 1}{p_3},$$

from which we get

$$(4) \qquad p_3 + 1 = \frac{2^{a_1+1} 3^{a_2+1}}{2^{a_1+1} + 3^{a_2+1} - 1}.$$

Hence the only prime factors that $p_3 + 1$ can have are 2 and 3:

$$p_3 + 1 = 2^{\mu} 3^{\nu}. \qquad (0 < \mu < a_1 + 1, \quad 0 \leqq \nu \leqq a_2 + 1).$$

Here, μ cannot have the value 0, since $p_3 + 1$ is an even number. From (3) we get

$$p_3 2^{a_1+1} 3^{a_2+1} = (2^{a_1+1} - 1)(3^{a_2+1} - 1)(p_3 + 1),$$

or

$$(5) \qquad p_3 2^{a_1+1-\mu} 3^{a_2+1-\nu} = (2^{a_1+1} - 1)(3^{a_2+1} - 1).$$

* Read before the Chicago Section of the American Mathematical Society, April 6, 1901.

† *cf.* Lehmer, *l. c.*

Since the parentheses in the right hand member of this equation are prime respectively to 2 and 3, two possible cases present themselves. We must have either

$$\text{I.}\quad \begin{cases} 3^{a_2+1} - 1 = p_3\, 2^{a_1+1-\mu} = 2^{a_1+1}\, 3^{\nu} - 2^{a_1+1-\mu} \\ 2^{a_1+1} - 1 = 3^{a_2+1-\nu} \end{cases}$$

or

$$\text{II.}\quad \begin{cases} 3^{a_2+1} - 1 = 2^{a_1+1-\mu} \\ 2^{a_1+1} - 1 = p_3\, 3^{a_2+1-\nu} = 2^{\mu}\, 3^{a_2+1} - 3^{a_2+1-\nu} \end{cases}$$

Case I. In this case we have

$$2^{a_1+1} - 1 = 3^{a_2+1-\nu}.$$

Let
$$a_1 + 1 = 2^{\lambda}(2n + 1).$$

Then
$$2^{a_1+1} - 1 = 2^{2^{\lambda}(2n+1)} - 1.$$

1. If $\lambda = 0$, then $2^{a_1+1} - 1 = 2 \cdot 4^n - 1 = 2(4^n - 1) + 1$ which is not divisible by 3, since $(4^n - 1)$ is divisible by 3.
2. If $\lambda = 1$, then $2^{a_1+1} - 1 = (2^{2n+1} - 1)(2^{2n+1} + 1)$, which contains other prime factors besides 3, unless $n = 0$; since if one of the factors is divisible by 3, the other is not. For $n = 0$ we have $a_1 + 1 = 2$ and $a_2 + 1 = \nu + 1$ and hence

$$3^{\nu+1} - 1 = 2^2\, 3^{\nu} - 2^{2-\mu},$$

or
$$3^{\nu} = 2^{2-\mu} - 1.$$

This gives $\mu = 1$, $\nu = 0$ which is impossible since $p_3 > 3$.

3. If $\lambda > 1$, then $2^{a_1+1} - 1 = 16^{2^{\lambda-2}(2n+1)} - 1$, which is always divisible by 5.

Thus in Case I there are no multiply perfect numbers of the type here considered.

Case II. In this case we have

$$3^{a_2+1} - 1 = 2^{a_1+1-\mu}.$$

Let
$$a_2 + 1 = 2^{\lambda}(2n + 1).$$

Then
$$3^{a_2+1} - 1 = 3^{2^{\lambda}(2n+1)} - 1.$$

1. If $\lambda = 0$, then $3^{a_2+1} - 1 = 3(9^n - 1) + 2$, which contains other primes besides 2 unless $n = 0$. For $n = 0$ we have $a_2 + 1 = 1$ which is impossible.

2. If $\lambda = 1$, then $3^{a_2+1} - 1 = 9(81^n - 1) + 8$, which contains other primes besides 2 unless $n = 0$. For $n = 0$ we have $a_2 + 1 = 2$ and $a_1 + 1 = \mu + 3$. Hence

$$2^{\mu+3} - 1 = 2^{\mu} . 3^2 - 3^{2-\nu}$$

or

$$2^{\mu} = 3^{2-\nu} - 1.$$

The only values of μ and ν which satisfy this equation are $\mu = 3$, $\nu = 0$ and $\mu = 1$, $\nu = 1$. The corresponding values of a_1, a_2, p_3 are $a_1 = 5$, $a_2 = 1$, $p_3 = 7$ and $a_1 = 3$, $a_2 = 1$, $p_3 = 5$.

It is found by trial (Lehmer, *l. c.*) that the resulting numbers, $2^5 \cdot 3 \cdot 7$ and $2^3 \cdot 3 \cdot 5$ are multiply perfect.

3. If $\lambda > 1$, then $3^{a_2+1} - 1 = 81^{2^{\lambda-2}(2n+1)} - 1$, which is divisible by 5.

Hence the only multiply perfect numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2^{a_2} p_3$ are the two numbers $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$.

PURDUE UNIVERSITY, FEBRUARY, 1901.

THE ISOPERIMETRICAL PROBLEM ON ANY SURFACE.

By J. K. Whittemore.

One of the most famous problems of mathematics is the so-called isoperimetrical problem in the plane: to find among closed curves of given perimeter that enclosing the greatest area. That the solution of this problem is the circle was known to Pappus* who wrote about 300 A. D.

The problem can be solved by the methods of the Calculus of Variations.† I give here a brief solution of the problem stated in a slightly different form. Let it be required to draw a curve of given length, L, between two given points of the X-axis $(a, 0)$, and $(b, 0)$, supposing

$$0 < b - a < L < \frac{\pi}{2}(b - a),$$

such that the area between the curve and the X-axis shall be a maximum. Analytically, this is to make the integral I a maximum, subject to the condition that the integral L shall have a given value, where

$$I = \int_a^b y\,dx, \qquad L = \int_a^b \sqrt{1 + y'^2}\,dx.$$

The method of the Calculus of Variations consists in forming the integral,

$$U = \int_a^b (y + \lambda \sqrt{1 + y'^2})\,dx,$$

where λ is an undetermined constant, and then requiring that the first variation of U shall vanish. The differential equation that thus results is the following:

$$1 - \lambda \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0. \tag{1}$$

The expression $\dfrac{d}{dx} \dfrac{y'}{\sqrt{1 + y'^2}}$ is the curvature of the curve $y = y(x)$.

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* William Thomson, *Popular Lectures and Addresses*, vol. 2, p. 578.

† See *e. g.* Jordan, *Cours d'analyse*, 2d ed., vol. 3, p. 497, where necessary conditions are obtained; Kneser, *Lehrbuch der Variationsrechnung*, pp. 123 and 136, where a complete solution is given.

Hence (1) tells us that the radius of curvature of the curve sought is constant and equal to λ. The solution is then a circle of radius λ, which may be determined so as to make the length of the arc, between the two points, equal to L.

A natural generalization is the same problem on any surface. This I shall now consider, proving that the curve sought is, at all points at which it has a tangent that turns continuously, of *constant geodesic curvature*. The necessary condition has long since been obtained; *cf.* for example, Knoblauch, *Krumme Flächen*, p. 251; Kneser, *l. c.*, p. 125, and Darboux, *Théorie des surfaces*, vol. 3, p. 151. It is my purpose here to give a treatment different from those which I have seen, which from the point of view of differential geometry is particularly simple.

We suppose given a surface and on it a curve connecting two of its points. Assume an arbitrary system of curvilinear coordinates (u, v) on the surface, and let the coordinates of the given points be (u_0, v_0) and (u_1, v_1) respectively: the equation of the given curve,

$$v = f(u).$$

The problem before us is *to find a curve, $v = \phi(u)$, joining the two given points (u_0, v_0) and (u_1, v_1), having a given length L, and such that the area of the portion of the surface between the two curves, $v = f(u)$ and $v = \phi(u)$, shall be a maximum.*

The geodesic curvature $1/\rho_\psi$ of a curve on the surface, $\psi(u, v) = 0$, is given by the following equation:*

$$\frac{1}{\rho_\psi} = \frac{1}{\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\frac{F\frac{\partial\psi}{\partial v} - G\frac{\partial\psi}{\partial u}}{\sqrt{E(\frac{\partial\psi}{\partial v})^2 - 2F\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v} + G(\frac{\partial\psi}{\partial u})^2}} + \frac{\partial}{\partial v}\frac{F\frac{\partial\psi}{\partial u} - E\frac{\partial\psi}{\partial v}}{\sqrt{E(\frac{\partial\psi}{\partial v})^2 - 2F\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v} + G(\frac{\partial\psi}{\partial u})^2}}\right\}$$

If for $\psi(u,v)$, we write $v - \phi(u)$, we have for the geodesic curvature

$$\frac{1}{\rho_g} = \frac{1}{\sqrt{EG - F^2}}\left\{\frac{\partial}{\partial u}\frac{F + G\phi'}{\sqrt{E + 2F\phi' + G\phi'^2}} - \frac{\partial}{\partial v}\frac{F\phi' + E}{\sqrt{E + 2F\phi' + G\phi'^2}}\right\}$$

* See Bianchi, *Differentialgeometrie*, p. 149.

In these equations E, F, and G are the coefficients of the linear element of the surface; that is to say for any curve drawn on the surface, we have

$$ds^2 = dx^2 + dy^2 + dz^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

The given length L of the curve $v = \phi(u)$ joining the two given points is given by the integral,

$$L = \int_{u_0}^{u_1} \sqrt{E + 2F\phi' + G\phi'^2}\,du$$

The area I which we wish to make a maximum is given by the double integral,

$$I = \int_{u_0}^{u_1}\int_{f(u)}^{\phi(u)} \sqrt{EG - F^2}\,dv\,du.$$

The unknown function $v = \phi(u)$ will be found by causing to vanish the first variation of the integral

$$U = \int_{u_0}^{u_1} \left\{ \int_{f(u)}^{v} \sqrt{EG - F^2}\,dv + \lambda \sqrt{E + 2Fv' + Gv'^2} \right\} du,$$

where λ is a constant.

The differential equation for $v = \phi(u)$ deduced from this condition in the Calculus of Variations is the following:

$$\sqrt{EG - F^2} + \lambda \frac{\dfrac{\partial E}{\partial v} + 2\dfrac{\partial F}{\partial v}v' + \dfrac{\partial G}{\partial v}v'^2}{2\sqrt{E + 2Fv' + Gv'^2}} - \lambda \frac{d}{du}\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} = 0. \quad (2)$$

From this equation we show that the curve $v = \phi(u)$ is of constant geodesic curvature. It is the simplification of this proof by means of a suitably chosen system of curvilinear coordinates that is the main point of this paper.

Consider the differential equation

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0.$$

This equation, being of the first order and of the second degree, may be replaced by two equations each of the first order and of the first degree. Let the general solutions of these two equations be

$$\alpha(u, v) = c, \qquad \beta(u, v) = c'.$$

The curves represented by these equations are the "minimal curves" of the surface. If we choose them as coordinate curves we shall have

$$ds^2 = 2\mathfrak{F}\,d\alpha\,d\beta.$$

Suppose by this change of coordinates the equation of the curve $v = \phi(u)$ becomes $\beta = \theta(\alpha)$. Then for the geodesic curvature we have

$$\frac{1}{\rho_g} = \frac{1}{i\mathfrak{F}\sqrt{2}}\left(\frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{F}}{\theta'}} - \frac{\partial}{\partial\beta}\sqrt{\mathfrak{F}\theta'}\right) \tag{3}$$

The differential equation (2) is of the same general form for any system of curvilinear coordinates. If we use α and β for curvilinear coordinates, it becomes:

$$i\mathfrak{F} + \lambda\sqrt{2\beta'}\,\frac{\partial\sqrt{\mathfrak{F}}}{\partial\beta} - \frac{\lambda}{\sqrt{2}}\,\frac{d}{d\alpha}\sqrt{\frac{\mathfrak{F}}{\beta'}} = 0. \tag{2'}$$

Here we may write

$$\frac{d}{d\alpha}\sqrt{\frac{\mathfrak{F}}{\beta'}} = \frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{F}}{\theta'(\alpha)}} + \theta'\frac{\partial}{\partial\beta}\sqrt{\frac{\mathfrak{F}}{\theta'(\alpha)}}$$

and thus (2′) assumes the form:

$$i\mathfrak{F} + \frac{\lambda}{\sqrt{2}}\frac{\partial}{\partial\beta}\sqrt{\mathfrak{F}\theta'} - \frac{\lambda}{\sqrt{2}}\frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{F}}{\theta'}} = 0,$$

where θ' is regarded, in the partial differentiations, as a function of α. From this equation it follows that

$$\frac{1}{\lambda} = \frac{1}{i\mathfrak{F}\sqrt{2}}\left(\frac{\partial}{\partial\alpha}\sqrt{\frac{\mathfrak{F}}{\theta'}} - \frac{\partial}{\partial\beta}\sqrt{\mathfrak{F}\theta'}\right),$$

and hence by the aid of (3) we infer that the curve $\beta = \theta(\alpha)$ has the property that

$$\rho_g = \lambda = \text{const.}$$

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ON A SURFACE OF THE SIXTH ORDER WHICH IS TOUCHED BY THE AXES OF ALL SCREWS RECIPROCAL TO THREE GIVEN SCREWS.*

By E. W. Hyde.

The term *screw* was introduced by R. S. Ball, in his book entitled "The Theory of Screws," and is applied to that geometric entity which represents completely the motion of a rigid body moving in any manner in space of three dimensions; a system of rotations and translations of a rigid body being at any instant equivalent to a single rotation about a definite axis, together with a translation along it.

A screw, representing the geometric properties of this motion, is defined by Ball as "a straight line in space" (the axis of the screw) "with which a definite linear magnitude termed the pitch is associated."

In an article by the present writer on "The Directional Theory of Screws,"† a screw was defined in the language of the Directional Calculus as the sum of a line-vector,‡ or "sect" (the *axis* of the screw), given in position as well as direction, and a plane-vector perpendicular to it. The sect corresponds to the above mentioned single rotation; the plane-vector to the translation. It was shown that, if S_1 and S_2 are screws, their product according to Grassmann's method is

$$S_1 S_2 = (a_1 + a_2) \cos \theta - \delta \sin \theta,$$

in which a_1 and a_2 are the respective magnitudes of the plane-vector parts, or in other words of the pitches, of the two screws; δ is the perpendicular distance between the axes of the screws, and θ is the angle between the axes. If this product is zero, the screws are said to be *reciprocal*.

In the article cited, it was further shown that the axes of the screws which are reciprocal to three given screws are the totality of the generators of one system of a skew conicoid (ruled quadric surface), which changes in accordance with the variation of a parameter involved in its equation.

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* Presented to the American Mathematical Society at its meeting, 27 April, 1901.

† Annals of Mathematics, vol. 4 (1888), p. 137.

‡ Grassmann's *Linienteil*.

The object of this paper is primarily to determine and discuss the envelope of this conicoid, which is a surface touched by the axes of all screws of the system, and thus to enable one to grasp more fully the nature of the system. The surface determined, however, possesses properties of an interesting character, aside from its relation to the system of screws.

Deduction of the Equation of the Envelope S.

1. The equation of the conicoid above referred to is given as equation 71 of the article cited, and is the following:

$$p(s_1 + z|\epsilon_1)\cdot(s_2 + z|\epsilon_2)\cdot(s_3 + z|\epsilon_3)p = 0, \tag{1}$$

in which p is a variable point, s_1, s_2, s_3 are screws, ϵ_1, ϵ_2, ϵ_3 are vectors, and z is the parameter. The expression $|\epsilon$ denotes a plane-vector perpendicular to ϵ, so that $s_1 + z|\epsilon_1 = s_1'$, say, is also a screw.

It will be convenient to change equation (1) into a different form, which we proceed to do. Take a fixed point e_0, and three mutually perpendicular unit vectors ι_1, ι_2, ι_3, and write

$$\begin{aligned} s_1 + z|\epsilon_1 &= e_0\iota_1 + (a_1 + z)|\iota_1 = e_0\iota_1 + z'|\iota_1, \\ s_2 + z|\epsilon_2 &= e_0\iota_2 + (a_2 + z)|\iota_2 = e_0\iota_2 + z''|\iota_2, \\ s_3 + z|\epsilon_3 &= e_0\iota_3 + (a_3 + z)|\iota_3 = e_0\iota_3 + z'''|\iota_3.^* \end{aligned}$$

Therefore equation (1) becomes

$$\begin{aligned} & p(e_0\iota_1 + z'|\iota_1)\cdot(e_0\iota_2 + z''|\iota_2)\cdot(e_0\iota_3 + z'''|\iota_3)p \\ &\quad = p(e_0\iota_1 + z'\iota_2\iota_3)\cdot(e_0\iota_2 + z''\iota_3\iota_1)\cdot(e_0\iota_3 + z'''\iota_1\iota_2)\, p \\ &\quad = - z'(e_0 p\iota_2\iota_3)^2 - z''(e_0 p\iota_3\iota_1)^2 - z'''(e_0 p\iota_1\iota_2)^2 - z'z''z''' = 0. \end{aligned}$$

Now change to a vector system by writing $p = e_0 + \rho$; whence $e_0 p\iota_1\iota_2 = e_0(e_0 + \rho)\iota_1\iota_2 = e_0\rho\iota_1\iota_2 = \rho|\iota_3$,† with similar values for the other terms; thus equation (1) becomes, on replacing z', z'', z''' by their values,

$$\begin{aligned} (a_1 + z)(\rho|\iota_1)^2 + (a_2 + z)(\rho|\iota_2)^2 + (a_3 + z)(\rho|\iota_3)^2 & \\ + (a_1 + z)(a_2 + z)(a_3 + z) = 0.‡ & \end{aligned} \tag{2}$$

---

* This apparently restricts the system of screws, s_1, s_2, s_3, by making them mutually perpendicular, and confluent; but it may be shown that *any* system reciprocal to three given screws has three screws belonging to it thus situated, and, since *any* three screws of the system determine it these three may be taken for that purpose. Our proceeding is therefore general.

† Equivalent to the quaternion expression $S\rho\iota_3$.

‡ This equation is identical with the last equation on p. 121 of Ball's *Theory of Screws*.

This is the vector equation of the conicoid; it involves the varying parameter z. The envelope of this conicoid will be a surface touched by the axes of all the screws reciprocal to s_1, s_2, s_3.

This equation may be transformed into an equation in Cartesian coordinates as follows. The quantities $\rho|\iota_1$, $\rho|\iota_2$, $\rho|\iota_3$ that enter in equation (2) are the scalar projections of ρ on the mutually perpendicular lines of reference, *i. e.* they are the ordinary Cartesian coordinates x, y, z. Replacing the parameter z by μ to prevent confusion we have as the equivalent of equation (2) the following:

$$(a_1 + \mu)\, x^2 + (a_2 + \mu)\, y^2 + (a_3 + \mu)\, z^2 + (a_1 + \mu)(a_2 + \mu)(a_3 + \mu) = 0 \quad (2')$$

as the equation of the conicoid in rectangular coordinates. Multiplying out and rearranging terms, this equation may be written in the form:

$$\mu^3 + \mu^2\, \Sigma a + \mu\, u + v = 0, \qquad (3)$$

in which

$$u = x^2 + y^2 + z^2 + \Sigma aa,$$
$$v = a_1 x^2 + a_2 y^2 + a_3 z^2 + a_1 a_2 a_3.$$
$$\Sigma a = a_1 + a_2 + a_3, \quad \Sigma aa = a_1 a_2 + a_2 a_3 + a_3 a_1.$$

Equation (2′) *or* (3) *is the Cartesian equation of the conicoid; it involves the varying parameter μ.* The envelope of this conicoid is found, as before, by eliminating μ between equation (3) and its partial derivative with respect to μ; it is

$$S = 4u^3 - (\Sigma a)^2 u^2 - 18uv\Sigma a + 27v^2 + 4(\Sigma a)^3 v = 0. \qquad (4)$$

Discussion of the Surface S.

2. *Discussion of Equation* (4). Since no terms of odd degree in x, y, or z appear in equation (4), *the surface S is symmetrical with respect to each of the three reference planes.* It is evident at once that S passes through the curve of intersection of the sphere $u = 0$ and the conicoid $v = 0$; also, as (4) can be written in the form

$$S = u' u^2 + v' v = 0 \qquad \text{where} \qquad u' = 4u - (\Sigma a)^2,$$

S is tangent to v along the curve in which v is cut by u; *i. e.* along the curve $u = 0$, $v = 0$, which we shall call the curve $u\,v$. This is plainly as it should be, because $v = 0$ is the surface of equation (3) when $\mu = 0$. S also passes through the points of intersection of $v = 0$ with the surface $u' = 0$.

3. *Expression of the Coefficients in Terms of the Differences of the a's.* It appears that if (4) be expanded by inserting the values of u and v and multiplying out, the literal coefficients will all be homogeneous functions of a_1, a_2, a_3, which are the roots of a cubic whose coefficients are Σa, Σaa and $a_1 a_2 a_3$. *These literal coefficients may then all be expressed in terms of the differences of these roots.* In fact, if we write $d_1 = a_2 - a_3$, $d_2 = a_3 - a_1$, $d_3 = a_1 - a_2$, we find that (4) may be written in the form

$$x^6 + y^6 + z^6 - \tfrac{1}{4} d_1^2 d_2^2 d_3^2 + 3x^4\left(y^2 + z^2 - \frac{d_1^2 + 8d_2d_3}{12}\right) + 3y^4\left(z^2 + x^2 - \frac{d_2^2 + 8d_3d_1}{12}\right)$$

$$+ 3z^4\left(x^2 + y^2 - \frac{d_3^2 + 8d_1d_2}{12}\right) + \tfrac{1}{2}\left[d_2d_3\,(d_1^2 + 2d_2d_3)\,x^2 + d_3d_1\,(d_2^2 + 2d_3d_1)\,y^2\right.$$

$$\left. + d_1d_2\,(d_3^2 + 2d_1d_2)\,z^2\right] + 6x^2y^2z^2 + \tfrac{1}{2}\left[\,(d_2d_3 - 10d_1^2)\,y^2z^2 + (d_3d_1 - 10d_2^2)\,z^2x^2\right.$$

$$\left. + (d_1d_2 - 10d_3^2)\,x^2y^2\right] = 0. \qquad (5)$$

4. *Simplification of the Equation of the Surface.* Equation (5) contains only *two* independent constants, since $d_1 + d_2 + d_3 = 0$. As it does not contain the a's, it is evident that any values may be assigned to them which do not alter the values of the d's, *i. e.* each a may be increased by any given number n,* which may be positive or negative. Let us write then

$$\left\{\begin{aligned} a_1' &= a_1 - a_2 = d_3 \\ a_2' &= a_2 - a_2 = 0 \\ a_3' &= a_3 - a_2 = - d_1 \end{aligned}\right\}; \therefore \left\{\begin{aligned} d_1 &= a_2' - a_3' = a_2 - a_3 \\ d_2 &= a_3' - a_1' = a_3 - a_1 \\ d_3 &= a_1' - a_2' = a_1 - a_2 \end{aligned}\right\}, \left\{\begin{aligned} \Sigma a' &= d_3 - d_1 \\ \Sigma a'a' &= - d_3 d_1 \\ a_1'a_2'a_3' &= 0 \end{aligned}\right\}.$$

Substituting these values in (4), it becomes

$$4u^3 - (d_3 - d_1)^2 u^2 - 18(d_3 - d_1)uv + 27v^2 + 4(d_3 - d_1)^3 v = 0, \qquad (6)$$

and we have

$$u = x^2 + y^2 + z^2 - d_3 d_1, \qquad (7)$$

$$v = d_3 x^2 - d_1 z^2 = (x\sqrt{d_3} + z\sqrt{d_1})\,(x\sqrt{d_3} - z\sqrt{d_1}). \qquad (8)$$

These are the simplest forms to which the equations can be reduced.

If we assume $a_1 > a_2 > a_3$, which does not affect the nature of S, but only its position relatively to the axes of reference, then d_1 and d_3 will always be positive, and $v = 0$ will represent two real planes through the Y axis. *S therefore touches the planes $v = 0$ along their circles of intersection with the sphere*

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* This is seen at once to be in agreement with the manner in which the parameter μ enters equation (2′).

$u=0$, *whose radius is* $\sqrt{d_3 d_1}$. These circles are shown in fig. 2 and are designated by the numerals 23 and 25. By assuming different values of a_1, a_2, a_3 such that the d's are unchanged, an infinite number of curves $u\,v$ can be obtained, along each of which S touches v.

5. *Intersections of v and S.* Let us eliminate z between $v=0$ and equation (8): then since $z^2 = \dfrac{d_3}{d_1}x^2$, equation (8) becomes

$$u^2[4\,u - (d_3 - d_1)^2] =$$

$$\left(y^2 - \frac{d_2}{d_1}x^2 - d_3 d_1\right)\left[4\,y^2 - \frac{4d_2}{d_1}x^2 - 4d_3 d_1 - (d_3 - d_1)^2\right] = 0,$$

or

$$\left(y^2 - \frac{d_2}{d_1}x^2 - d_3 d_1\right)^2 = 0, \tag{9}$$

and

$$y^2 - \frac{d_2}{d_1}x^2 - \tfrac{1}{4}d_2^2 = 0. \tag{10}$$

Hence the two planes $v=0$ are tangent to S along the curves whose projections on $X\,Y$ are the ellipse of (9), *and intersect S in the curves whose projections on $X\,Y$ are the ellipse of* (10). These are ellipses because d_2 is negative while d_3 and d_1 are positive. *These curves of section of v and S are in fact circles*, since (9) and (10) are simply the equations of the projections of the curves of section of v with u and u' as in §2, *i. e.* of spheres and planes. The circles of equation (10) are shown in fig. 2 as 14 and 16. In figs. 1 and 3 they coincide with the circles of equation (9).

6. *Intercepts of S on the Axes.* In equation (5) let $y = z = 0$, then

$$\begin{aligned} x^6 - \tfrac{1}{4}(d_1^2 + 8d_2 d_3)x^4 + \tfrac{1}{2}d_2 d_3(d_1^2 + 2d_2 d_3)x^2 - \tfrac{1}{4}\,d_1^2 d_2^2 d_3^2 \\ = (x^2 - d_2 d_3)^2(x^2 - d_1^2) = 0. \end{aligned}$$

By symmetry we have at once similar results when $z = x = 0$, and when $x = y = 0$, hence we have for the intercepts on the axes of reference,

$$\left.\begin{aligned} x_0 &= \pm\sqrt{d_2 d_3} \quad\text{and}\quad x_0' = \pm\tfrac{1}{2}\,d_1, \\ y_0 &= \pm\sqrt{d_3 d_1} \quad\text{and}\quad y_0' = \pm\tfrac{1}{2}\,d_2, \\ z_0 &= \pm\sqrt{d_1 d_2} \quad\text{and}\quad z_0' = \pm\tfrac{1}{2}\,d_3. \end{aligned}\right\} \tag{11}$$

7. *Nodal Points. The values of x_0, y_0, z_0*, being derived from square factors, *correspond to double points in the curves of section of S with the reference planes, and to nodes of S, real or imaginary.* Of the six values only two

are real, because d_2 is negative, namely $y_0 = \pm\sqrt{d_3 d_1}$. The intercepts x_0', y_0', z_0' are always real.

8. *Nodal Tangent Cones.* To find the equations of the nodal tangent cones at $(0, +\sqrt{d_3 d_1}, 0)$ and $(0, -\sqrt{d_3 d_1}, 0)$. Change the origin to the first point by writing $y = y' + \sqrt{d_3 d_1}$: then we have $u = x^2 + y'^2 + 2y'\sqrt{d_3 d_1} + z^2$, while v is unchanged, u and v being taken as in (7) and (8). The equation of the tangent cone at the new origin can be obtained by writing the terms of the second degree in the transformed S equal to zero. We get

$$4(d_3 - d_1)^3(d_3 x^2 - d_1 z^2) - (d_3 - d_1)^2(4 d_3 d_1 y'^2) = 0,$$

or

$$(d_3 - d_1)(d_3 x^2 - d_1 z^2) - d_3 d_1 y'^2 = 0.$$

This is the equation of the tangent cone when the origin is at the node. Taking the origin at the centre of S we have for the two nodal cones

$$(d_3 - d_1)(d_3 x^2 - d_1 z^2) - d_3 d_1 (y \mp \sqrt{d_3 d_1})^2 = 0, \tag{12}$$

the upper sign giving the one on the positive side of the origin. If we let $x = 0$ in (12), we have for the *nodal tangents in the YZ plane*

$$y = \pm z\sqrt{\frac{d_1 - d_3}{d_3}} \pm \sqrt{d_3 d_1}; \tag{13}$$

and similarly, putting $z = 0$, we have for the *nodal tangents in the XY plane*

$$y = \pm x\sqrt{\frac{d_3 - d_1}{d_1}} \pm \sqrt{d_3 d_1}. \tag{14}$$

Since we are assuming d_3 and d_1 both positive, one of these sets of tangents will be real, while the other set will be imaginary. In fig. 2, which is constructed with the values $d_1 = 4$, $d_3 = 1$, the lines of (13) are real, and those of (14) imaginary, so that as regards the section by the XY plane $(0, +\sqrt{d_3 d_1}, 0)$ and $(0, -\sqrt{d_3 d_1}, 0)$ are conjugate points.

9. *Determination of the Cuspidal Edges, or Edges of Regression, of the Surface.* If we assume given constant values for d_1, d_2 and d_3, and let the a's vary, the uv line will generate the surface. The cusp line, as the figures show, is the locus of points on S most distant from the Z axis. We will take the point where the plane $y = mx$ cuts the uv curve, find its x coordinate in terms of one of the a's, say a_3, and the d's: then, taking a as an independent variable, determine its value when x^2 is a maximum. Since $a_1 = a_3 - d_2$ and $a_2 = a_3 + d_1$, we have

$$\Sigma aa = a_2a_3 + a_3a_1 + a_1a_2 = (a_3 + d_1)\,a_3 + a_3\,(a_3 - d_2) + (a_3 - d_2)(a_3 + d_1)$$
$$= 3a_3^2 + 2a_3\,(d_1 - d_2) - d_1d_2 = 3a_3^2 + 2a_3\delta_3 - d_1d_2\,;$$

in which we have written $d_1 - d_2 = \delta_3$. We shall also find it convenient to use the relations

$$d_2 - d = \delta_1 \quad \text{and} \quad d_3 - d_1 = \delta\,.$$

We have also

$$a_1a_2a_3 = (a_3 - d_2)\,(a_3 + d_1)\,a_3 = a_3\,(a_3^2 + a_3\delta_3 - d_1d_2).$$

With these values we have, putting mx for y,

$$a_3u = a_3x^2 + a_3m^2x^2 + a_3z^2 + a_3\,(3a_3^2 + 2a_3\delta_3 - d_1d_2) = 0,$$
$$v = (a_3 - d_2)\,x^2 + (a_3 + d_1)m^2x^2 + a_3z^2 + a_3\,(a_3^2 + a_3\delta_3 - d_1d_2) = 0.$$
$$\therefore\ v - a_3u = (m^2d_1 - d_2)x^2 - a_3\,(2a_3^2 + a_3\delta_3) = 0\,;$$
$$\therefore \qquad (m^2d_1 - d_2)\,\frac{d}{da_3}\,x^2 = 6a_3^2 + 2\,a_3\delta_3 = 0.$$

Thus $\qquad a = 0 \quad \text{and} \quad a_3 = -\tfrac{1}{3}\,\delta_3.$

Differentiating again

$$\frac{d^2}{da_3^2}\,x^2 = \frac{12\,a_3 + 2\delta_3}{m^2d_1 - d}\,.$$

When $a_3 = 0$ the second derivative is positive, because $\delta_3 = d_1 - d_2$, and d_2 is negative, while d_1 is positive, indicating a minimum; and when $a_3 = -\tfrac{1}{3}\delta_3$ the second derivative is negative, indicating a maximum. Inserting the value $a_3 = -\tfrac{1}{3}\delta_3$ we find the maximum value of x^2 to be

$$x_m^2 = \frac{\delta_3^3}{27\,(m^2d_1 - d_2)}\,.$$

We find also

$$a_1 = a_3 - d_2 = -\tfrac{1}{3}\delta_3 - d_2 = -\tfrac{1}{3}d_1 + \tfrac{1}{3}d_2 - d_2$$
$$= -\tfrac{1}{3}(d_1 + 2d_2) = -\tfrac{1}{3}(2d_2 - d_2 - d_3) = -\tfrac{1}{3}\delta_1,$$

and similarly $a_2 = -\tfrac{1}{3}\delta_2$. Hence

$$u_c = x^2 + y^2 + z^2 + \tfrac{1}{9}\,\Sigma\,\delta\,\delta = x^2 + y^2 + z^2 + \tfrac{1}{3}\Sigma d\,d = 0, \tag{15}$$
$$-\,3\,v_c = \delta_1x^2 + \delta_2y^2 + \delta_3z^2 + \tfrac{1}{9}\,\delta_1\,\delta_2\,\delta_3 = 0. \tag{16}$$

The surface S is tangent to v_c along the cusp curve $u_c\,v_c$, and hence the cusp tangent at any point is easily found.

10. *Various Forms of S.* We will now consider the different forms that S may assume through change of the constants. As appears by equations

(6), (7), (8), the surface depends on only two constants, d_3 and d_1, which are always positive; hence, its form depends only on their relative magnitude. We will consider three cases.

Case 1. Let $d_3 = d_1$; then equation (6) reduces to

$$4u^3 + 27v^2 = 4(x^2 + y^2 + z^2 - d_1^2)^3 + 27\,d_1^2(x^2 - z^2)^2 = 0. \tag{17}$$

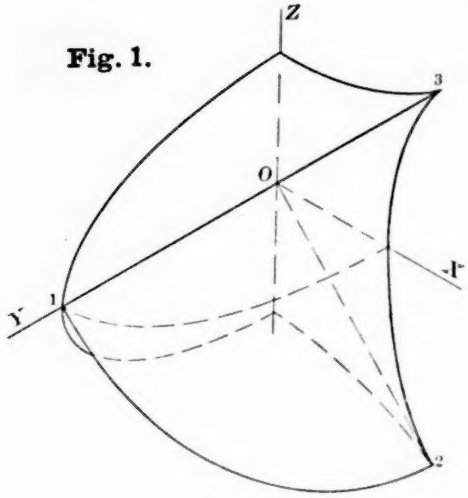

FIG. 1.

A fourth part of the surface is shown in fig. 1, which, like the other two, is drawn in isometric projection. Equation (17) shows at once that *S can have real points only when the radius vector is less than, or equal to, d_1; hence the surface is wholly within or upon the sphere $u = 0$, and the cusp lines are great circles of that sphere, whose planes bisect the angles between the XY and YZ planes*; for when $d_3 = d_1$,

$$u_c = x^2 + y^2 + z^2 - d_1^2 = 0,$$

and

$$v_c = d_1(x^2 - z^2) = 0 = (x - z)(x + z).$$

The section of S by the ZX plane is the hypocycloid described when the radius of the rolling circle is one fourth that of the fixed circle, and having the equation

$$x^{2/3} + z^{2/3} = d_1^{2/3}$$

when referred to axes bisecting the angles between the axes of Z and X in the

figure. The nodal tangent cones, equation (12), reduce to double planes parallel to the $Z\,X$ plane, and touching S at $(0, d_1, 0)$ and $(0, -d_1, 0)$.

Case 2. When $1 < d_1/d_3 < \infty$ we have the general case, an example of

Fig. 2.

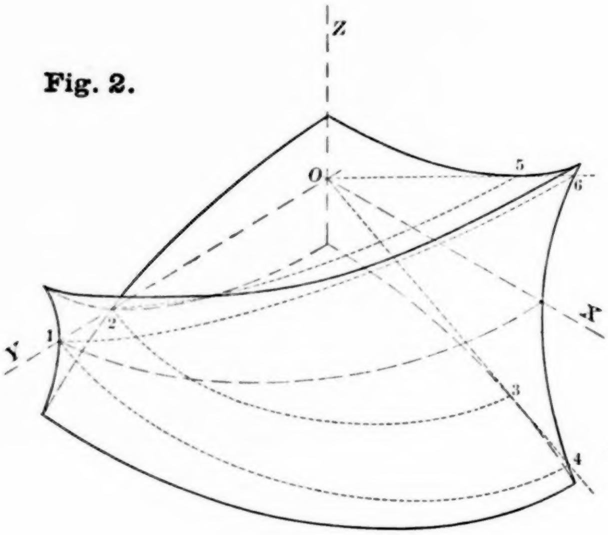

FIG. 2.

which is shown in fig. 2, for which $d_1 = 4$ and $d_3 = 1$. The planes $v = 0$ pass through $O\,Y$ and the lines O5 and O3 respectively and touch S along the circles 25 and 23. There are nodes at $(0, 2, 0)$ and $(0, -2, 0)$, only the former being shown. The equation of the nodal tangent cones is

$$3(x^2 - 4z^2) - 4(y \mp 2)^2 = 0.$$

If d_3 approaches d_1 in value the circles of equations (9) and (10) (in fig. 2 25 and 16, and 23 and 14) approach each other, finally coinciding. The nodal cone flattens out perpendicularly to $O\,Y$, and reduces finally to a double plane as we saw above.

If, on the other hand, d_3 diminishes toward zero, we approximate toward the third case as shown in fig. 3.

Case 3. When $d_3 = 0$ the surface becomes one of revolution about the

Fig. 3.

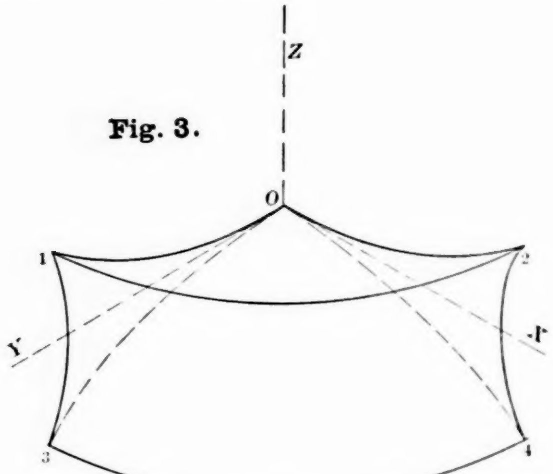

FIG. 3.

Z axis, the cusp curves are horizontal circles given by the equations

$$\left.\begin{aligned} x^2 + y + z^2 &= \tfrac{1}{3} d_1^2 \\ x^2 + y^2 - 2z^2 &= \tfrac{2}{9} d_1^2 \end{aligned}\right\},$$

and the nodes have receded to the centre and disappeared, the equation of the nodal cone becoming $z^2 = 0$.

11. *Conclusion.* The surface S, which we have determined and discussed, assists the mind in forming a conception of the screw system reciprocal to s_1, s_2, s_3. *Every screw axis of the system touches S*, but *not* every line that touches S belongs to the system; for, in order to do so, it must also be a generator of some one of the skew hyperboloids of the system formed by assigning different values to μ in the equation

$$(a_1 + \mu)x^2 + (a_2 + \mu)y^2 + (a_3 + \mu)z^2 + (a_1 + \mu)(a_2 + \mu)(a_3 + \mu) = 0.$$

The surface S shows us how these hyperboloids are situated and how they vary.

CINCINNATI, O., MAY, 1901.

NOTE SUR L'ÉVALUATION D'UNE INTÉGRALE DÉFINIE.

(A propos de la Note de M. Pell, ANNALS OF MATHEMATICS (2), t. 1, no. 3.)

PAR M. D. SINTSOF.

M. Pell dans sa note : "Evaluation of a definite integral" — ANNALS (2) t. 1, no. 3 — s'occupe de l'intégrale

$$\int_0^\infty e^{-px^2-qx^{-2}} \begin{matrix}\sin\\ \cos\end{matrix} (rx^2 + sx^{-2})dx,$$

qui selon Bierens de Haan a été évaluée par M. P. Helmling. L'ouvrage de P. Helmling "Transformation und Ausmittelung bestimmter Integrale mit besonderer Rücksicht auf grössere Werthe der Gränzen und implicirten Constanten" parut à Milan en 1854 et parait être assez rare à présent. Il ne sera donc pas inutile d'indiquer ici brièvement le procédé de P. Helmling (*l. c.*, p. 87).

Soit
$$y = \int_0^\infty e^{-r_1^2 x^2 e^{2\mu i} - r^2 x^{-2} e^{2\nu i}}\, dx \tag{1}$$

l'intégrale à évaluer. Différentions-la par rapport à r sous le signe de l'intégrale :

$$\frac{dy}{dr} = -\, 2re^{2\nu i}\int_0^\infty e^{-r_1^2 x^2 e^{2\mu i} - r^2 x^{-2} e^{2\nu i}}\, x^{-2}\, dx. \tag{a}$$

Introduisant ici au lieu de x la variable nouvelle x' :

$$x = \frac{r}{r_1} e^{(\nu-\mu)i} \cdot \frac{1}{x}, \tag{b}$$

P. Helmling conclut :

$$\frac{dy}{dr} = -\, 2r_1\, e^{(\mu+\nu)i}\, y. \tag{2}$$

L'intégration de cette équation différentielle entre y et r donne

$$y = c \cdot e^{-2rr_1 e^{(\mu+\nu)i}}.$$

La constante c peut être déterminée si l'on pose $r = 0$. Dans ce cas

$$(y)_{r=0} = c = \int_0^\infty e^{-r^2 x^2 e^{2\mu i}}\, dx = \frac{\sqrt{\pi}}{2r_1}\, e^{-\mu i}$$

de sorte qu'enfin

$$y = \frac{\sqrt{\pi}}{2r_1} e^{-(2rr_1 e^{(\mu+\nu)i} + \mu i)}. \tag{3}$$

Posons ici

$$r_1^2 e^{2\mu i} = \alpha + \beta i, \qquad r^2 e^{2\nu i} = \gamma + \delta i$$

et séparons la partie réelle et la partie imaginaire des deux expressions (1) et (3) de y. Nous aurons les formules finales de P. Helmling.

Tel est son procédé. Tout simple qu'il est, il donne toutefois lieu à deux remarques. Il repose sur les opérations (a) et (b), qu'il faut prouver.

(a) En premier lieu, la différentiation sous le signe de l'intégrale définie est une opération assez délicate et peut mener à des résultats illusoires si la fonction sous le signe d'intégrale devient infinie entre les limites d'intégration, ou si les limites mêmes sont infinies. Nous sommes dans le second cas, la fonction à intégrer

$$f(x, r) = e^{-r_1^2 x^2 e^{2\mu i} - r^2 x^{-2} e^{2\nu i}}$$

reste finie entre les limites $(0, \infty)$; elle devient nulle aux limites, si l'on a

$$\cos 2\mu > 0, \qquad \cos 2\nu > 0. \tag{4}$$

Ce sont bien les conditions indiquées par M. Pell, *l. c.*

On peut s'assurer aisément que dans ces conditions la différentiation est légitime. Posons en effet

$$f(x, r) = U - iV,$$

U et V étant deux fonctions réelles des variables réelles x et r remplissent les conditions de M. Picard (Traité d'analyse, t. 1, p. 33) : 1° pour x positif et suffisamment grand U et V ainsi que les dérivées

$$U'_r = -\frac{2r}{x^2}[U \cos 2\nu + V \sin 2\nu], \quad V'_r = +\frac{2r}{x^2}[U \sin 2\nu - V \cos 2\nu]$$

deviennent et restent plus petites en valeur absolue que Mx^{-n}, quelque grand que soit le nombre positif n ; 2° les expressions $U'_r(x, r + h_1) - U'_r(x, r)$ et

$V'_r(x, r + h_1) - V'_r(x, r)$ sont nulles pour $x = 0$, et pour toute autre valeur positive de x on peut trouver h_1 tel que les dites différences soient aussi petites que l'on veut. Donc, comme le démontre M. Picard, nous pouvons écrire :

$$\frac{d}{dr}\int_0^\infty U\,dx = \int_0^\infty U'_r\,dx \quad ; \quad \frac{d}{dr}\int_0^\infty V\,dx = \int_0^\infty V'_r\,dx,$$

d'où

$$\frac{d}{dr}\int_0^\infty (U - iV)\,dx = \int_0^\infty (U'_r - iV'_r)\,dx \quad , \qquad q.\ e.\ d.^*$$

(*b*) L'introduction de la nouvelle variable x' par (*b*) en (*a*) donne, à proprement parler, non pas l'équation (**2**), mais l'égalité :

$$\frac{dy}{dr} = -\,2r_1 e^{(\mu+\nu)i}\int_0^\infty f(x', r)\,dx',$$

---

* On peut démontrer (*a*) directement. Divisons le champ d'intégration en deux : (0, 1), (1, ∞). L'intégrale $F(r) = \int_1^\infty f(x, r)dx$ donne :

$$\frac{F(r + \Delta r) - F(r)}{\Delta r} = \int_1^l \frac{f(x, r + \Delta r) - f(x, r)}{\Delta r}\,dx + \int_l^\infty \frac{f(x, r + \Delta r) - f(x, r)}{\Delta r}\,dx.$$

Quand Δr tend vers zéro, la première tend vers $\int_1^l f'_r(x, r)\,dx$, l'intervalle d'intégration étant finie, et la fonction $f(x, r)$ sans discontinuités. La seconde $\int_l^\infty \frac{f(x, r + \Delta r) - f(x, r)}{\Delta r}\,dx$ sera égale à $\int_l^\infty \lambda\, f'_r(x, r + \theta\Delta r)\,dx$, où λ reste fini. d'après la formule connue de M. Darboux (*Journal de mathématiques* 3e sér. t. 2. 1876) et pour $l = \infty$ l'intégrale tend donc vers zéro, comme l'on conclut avec M. Picard (*l. c.*), $f'_r(x, r)$ étant en valeur absolue moindre que Mx^{-n} ; donc vraiment

$$F'(r) = \int_1^\infty f'_r(x, r)\,dx.$$

En introduisant $x = 1/x'$, nous aurons

$$P(r) = \int_0^1 f(x, r)\,dx = \int_1^\infty f\left(\frac{1}{x'}, r\right)\frac{dx'}{x'^2}$$

et nous démontrerons de la même manière que

$$P'(r) = \int_1^\infty f'_r\left(\frac{1}{x'}, r\right)\frac{dx'}{x'^2} = \int_0^1 f'_r(x, r)dx ;$$

donc enfin

$$\frac{d}{dr}\int_0^\infty f(x, r)dx = \int_0^\infty f'_r(x, r)\,dx, \qquad q.\ e.\ d.$$

Cette démonstration est due à M. Osgood.

où l'intégrale à droite est prise suivant la droite qui fait l'angle $\nu-\mu$ (comprise entre $-\frac{\pi}{2}$ et $+\frac{\pi}{2}$) avec l'axe des abscisses. Or cette intégrale est bien égale à y, ce que l'on voit par exemple en prenant l'intégrale suivant le contour fermé formé par la partie positive de l'axe des abscisses, le quart de cercle du rayon infiniment grand et la droite mentionnée plus haut — l'intégrale suivant l'arc du cercle étant nulle, les deux intégrales sont bien égales.

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